6.3 Undetermined Coefficients and the Annihilator Method

Notation

An $n^{th}$-order differential equation can be written as

$$a_n D^n y + a_{n-1} D^{n-1} y + \ldots + a_1 D y + a_0 y = f(x)$$

where $D^k y = \frac{d^k y}{dx^k}$; $k = 0, 1, 2, \ldots, n$

It can also be written even more simply as

$$L[y] = f(x)$$

where $L$ denotes the linear $n^{th}$-order differential operator or characteristic polynomial

$$a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0$$

In this section, we will look for an appropriate linear differential operator that annihilates $f(x)$.

Factoring Operators

Example

1. Rewrite the differential equation using operator notation and factor.

   $$y'' + 4y' + 4y = 0$$

   $$(D^2 + 4D + 4)y = 0$$

   $$(D + 2)(D + 2)y = 0$$

   $$(D + 2)^2 y = 0$$
Annihilator Operators

If $L$ is a linear differential operator with constant coefficients and $f$ is a sufficiently differentiable function such that

$$L[f(x)] = 0$$

then $L$ is said to be an annihilator of the function.

Examples – Find the differential operator that annihilates each function.

2. $y = k$

$$D[k] = 0$$

3. $y = x$

$$D^2[x] = D[1] = 0$$

4. $y = x^4$

$$D\left[x^4\right] = 4x^3 \quad D^3\left[x^4\right] = 24x \quad D^5\left[x^4\right] = 0$$

The function $x^n$ is annihilated by the differential operator

$$D^{n+1}$$

Note: This annihilator will also annihilate any linear combination of these functions:

$$c_{n-1}x^n + c_nx^{n-1} + \cdots + c_1x + c_0$$

The function $x^n e^{\alpha x}$ is annihilated by the differential operator

$$(D - \alpha)^{n+1}$$

If $L_1$ annihilates $y_1(x)$ and $L_2$ annihilates $y_2(x)$, then the product of the differential operators $L_1 \cdot L_2$ will annihilate the linear combination

$$c_1y_1 + c_2y_2$$

Note: The differential operator that annihilates a function is not unique.
Example

5. Verify that the given differential operator annihilates the indicated function

\[ (2D - 1)[4e^{x/2}] = 2D[4e^{x/2}] - 4e^{x/2} \]
\[ = 2(2e^{x/2}) - 4e^{x/2} \]
\[ = 0 \]

Find a differential operator that annihilates the given function.

6. \[ x^3 - 5x^4 \]
   \[ n = 4 \]
   \[ \boxed{D^5} \]

7. \[ 1 + 7e^{2x} \]
   \[ n = 0 \]
   \[ \alpha = 2 \]
   \[ \boxed{D(D - 2)} \]
   \[ D(D - 2)[1 + 7e^{2x}] = (D^2 - 2D)[1 + 7e^{2x}] \]
   \[ = D^2[1] + D^2[7e^{2x}] - 2D[1] - 2D[7e^{2x}] \]
   \[ = 0 + D[14e^{2x}] - 2(0) - 2(14e^{2x}) \]
   \[ = 28e^{2x} - 28e^{2x} \]
   \[ = 0 \]

Note: \[ (D - 2)[1 + 7e^{2x}] = D[1] + D[7e^{2x}] - 2(1) - 2(7e^{2x}) \]
\[ = 0 + 14e^{2x} - 2 - 14e^{2x} \]
\[ = -2 \] Doesn't work

8. \[ 2e^{-3x} - 3xe^{-3x} \]
   \[ \alpha = -3 \]
   \[ \boxed{(D + 3)^2} \]
\[ x e^{-2x} + e^{-2x} \]
\[ \boxed{(D + 3)^2 (D + 2)} \]
The functions \( x^n e^{ax} \cos \beta x \) and \( x^n e^{ax} \sin \beta x \) are annihilated by the differential operator
\[
[D^2 - 2\alpha D + \alpha^2 + \beta^2]^{n+1}
\]

Special Case: when \( \alpha = 0 \) and \( n = 0 \), then
\[
(D^2 + \beta^2) \{\cos \beta x \} \sin \beta x = 0
\]

**Example**

9. \( 1 + \sin x \) \quad \text{Annihilator } D \left( D^2 + 1 \right)
   \begin{align*}
   n &= 0 \\
   \alpha &= 0 \\
   \beta &= 1
   \end{align*}

10. \( 8x - \sin x + 10 \cos 5x \) \quad \text{Annihilator } D^2 \left( D^2 + 1 \right) \left( D^2 + 25 \right)
   \begin{align*}
   n &= 0 \\
   \alpha &= 0 \\
   \beta &= 5
   \end{align*}

**Method of Undetermined Coefficients: Annihilator**

1. Solve the associated homogeneous equation to find \( y_c \).
2. Find the annihilator for \( g(x) \) and apply it to both sides of the differential equation.
3. Solve the now homogeneous DE to find the general solution.
4. Identify the particular solution, \( y_p \), and find its derivatives.
5. Plug \( y_p \) and its derivatives into the original equation to find the unknown constant.

*Weakness* - must have constant coefficients and we must be able to find an annihilator for \( g(x) \).
Examples – Solve
11. \(y'' - 3y' - 4y = 3e^{2x}\)

First solve for \(y_h\):

\[ y'' - 3y' - 4y = 0 \]

Let \(y = e^{rx}\) then

\[ e^{rx}(r^2 - 3r - 4) = 0 \]

\[(r - 4)(r + 1) = 0 \]

\[ r = 4, -1 \]

\[ y_h = c_1 e^{4x} + c_2 e^{-x} \]

Second, find the annihilator for \(3e^{2x}\) : \((D - 2)\)

Apply to both sides of the original DE

\[(D - 2)(D^2 - 3D - 4)y = (D - 2)[3e^{2x}] \]

\[(D - 2)(D^2 - 3D - 4)y = 0 \]

Let \(y = e^{rx}\) then

\[ e^{rx}(r - 2)(r^2 - 3r - 4) = 0 \]

\[(r - 2)(r - 4)(r + 1) = 0 \]

\[ r = 2, 4, -1 \]

\[ y = c_1 e^{2x} + c_2 e^{4x} + c_3 e^{-x} \]

To solve for \(A\):

\[ y_p = Ae^{2x} \]

\[ y_p' = 2Ae^{2x} \]

\[ y_p'' = 4Ae^{2x} \]

Substituting into the original DE

\[ 4Ae^{2x} - 3(2Ae^{2x}) - 4Ae^{2x} = 3e^{2x} \]

\[ -6Ae^{2x} = 3e^{2x} \]

\[ A = -\frac{1}{2} \]

Thus the general solution is

\[ y = -\frac{1}{2} e^{2x} + c_1 e^{4x} + c_2 e^{-x} \]
12. \( y'' + 4y = 4\cos x + 3\sin x - 8 \)

Solve \( y'' + 4y = 0 \) Let \( y = e^{rx} \)

\[
e^{rx}(r^2 + 4) = 0
\]

\( r = \pm 2i \quad \Rightarrow \quad y_h = c_1 \cos 2x + c_2 \sin 2x
\)

Annihilator: \( D(D^2 + 1) \)

\( D(D^2 + 1)(D^2 + 4)y = 0 \)

Let \( y = e^{rx} \) Then

\[
e^{rx} r(r^2+1)(r^2+4) = 0 \quad \Rightarrow \quad r = 0, \pm i, \pm 2i
\]

\[
y = C_1 + C_2 \cos x + C_3 \sin x + C_4 \cos 2x + C_5 \sin 2x
\]

\( y = A + B \cos x + C \sin x \)

\( y' = -B \sin x + C \cos x \)

\( y'' = -B \cos x - C \sin x \)

Substituting into the original DE:

\[
-4A \cos x - 4C \sin x + 4A + 4B \cos x + 4C \sin x = 4 \cos x + 3 \sin x - 8
\]

\[
4A + 3B \cos x + 3C \sin x = 4 \cos x + 3 \sin x - 8
\]

\( 4A = -8 \quad 3B = 4 \quad 3C = 3 \)

\( A = -2 \quad B = \frac{4}{3} \quad C = 1 \)

\[
y = -2 + \frac{4}{3} \cos x + \sin x + c_1 \cos 2x + c_2 \sin 2x
\]